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Visual Interpolation of Data

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Abstract—Following a precise definition of shape-preserving interpolating functions to data, we construct in a new and elementary manner such a cubic spline $S_2 \in \mathcal{C}^2$, letting two additional knots per interval. We give an explicit description of S_2, which has satisfactory properties in all the aspects. All the results in the paper—obtained with elementary calculus—are based on the behavior of the derivative of any smooth interpolating function to these data.

Keywords—Cubic splines, Shape-preserving interpolating function.

1. INTRODUCTION

Let

$$\Pi_1 := (x_1, y_1), \dots, \Pi_n := (x_n, y_n),$$

be n points in $\mathbb{R}^2$, $n \geq 3$, where $x_1 < x_2 < \cdots < x_n$. Let $J := [x_1, x_n]$ and $J_i := [x_i, x_{i+1}]$, $1 \leq i \leq n-1$. We say that $S_1 \in \mathcal{C}^1(J)$ is an *interpolating* function (with respect to the previous data) if $S_1(x_i) = y_i$, $1 \leq i \leq n$. Our aim is to find an interpolating cubic spline $S_2 \in \mathcal{C}^2(J)$, with knots at x_i, $2 \leq i \leq n-1$, and further under the following conditions.

(i) S_2 is obtained locally on each J_i, depending generally on Π_{i-1}, Π_i, Π_{i+1}, Π_{i+2}, and in some cases also on Π_{i-2} and/or Π_{i+3}, whenever they exist.

(ii) S_2 is shape-preserving. We give below a precise definition of this concept but, roughly speaking, it means that S_2 preserves *at the same time* the monotone and convex character of the data.

(iii) If $f \in \mathcal{C}^3(J)$ is an interpolating function, then

$$\max_{x \in J} |f(x) - S_2(x)| \leq C\,K\,h^3,$$

where $K := \max |f'''|$ on J, $h := \max_{1 \leq i < n}\{x_{i+1} - x_i\}$, and C is a constant independent of f and of the data.

(iv) There are no restrictions on the behaviour of the data, and the computation of S_2 is straightforward and user-friendly.

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We indeed construct such a spline S_2, provided we let S_2 have two additional knots in each J_i. We wish to emphasize that the techniques to obtain S_2 are based on elements of euclidean geometry in the plane, and hence, elementary tools of calculus are only applied. It is in this sense in which the word 'visual' is used in the title. By this way the evidence that S_2 becomes shape-preserving is completely clear. Other important advantage of the method is that the selection of the additional knots is direct, i.e., no test is needed to see whether S_2 becomes shape-preserving.

Accordingly, we present here a theoretically self-contained procedure satisfying a set of properties which—up to our knowledge—have not simultaneously achieved in the previous related papers. We only mention the more recent works [2–5]. Other references can be seen therein.

2. THE THEORETICAL SUPPORT

We start with the broken line $S_0 \in C^0(J)$, which joins all the points Π_i by line segments. Consider its right derivative $S_0'(x)$, $x \in [x_1, x_n)$. The step function $S_0' : J \mapsto \mathbb{R}(S_0'(x_n) = S_0'(x_n^-))$, will be a permanent reference in what follows. We say that a right continuous function, $g : J \mapsto \mathbb{R}$, has r *strong sign changes* on J, $r = 1, 2, \ldots$, if there exist $\xi_j \in J$, $x_1 = \xi_0 < \xi_1 \cdots < \xi_{r+1} = x_n$, such that $g(x)g(x^*) \leq 0$ for $x \in [\xi_j, \xi_{j+1})$, $x^* \in [\xi_{j+1}, \xi_{j+2})$, and further $g(x_0)g(x_0^*) < 0$ for some $x_0 \in [\xi_j, \xi_{j+1})$ and some $x_0^* \in [\xi_{j+1}, \xi_{j+2})$, $0 \leq j \leq r-1$. We say that such a function g has 0 strong sign changes on J if $g(x) \geq 0$, or $g(x) \leq 0$, for $x \in [x_1, x_n)$. A function g_0 is said to be *increasing* (*decreasing*) on an interval if $g_0(z_0) \leq g_0(z_0^*)$ $(g_0(z_0) \geq g_0(z_0^*))$ for z_0, z_0^* in that interval, $z_0 < z_0^*$. Let $Q \in C^0(J)$ be the broken line on J that joins the points (m_i, d_i) by line segments, $1 \leq i \leq n-1$, where $m_i := (x_i + x_{i+1})/2$, $d_i := S_0'(m_i) = S_0'(x_i)$, extended on J without knots in $J_1 \cup J_{n-1}$.

DEFINITION. *We say that an interpolating function S_1 is piecewise monotone (briefly, PM) if S_1' has as many strong sign changes as S_0' on J. It is piecewise convex-concave (PC) if*

(PC$_1$) *for $2 \leq i \leq n-2$, if Q is strictly increasing, or strictly decreasing, on J_i, then S_1' is strictly increasing, or strictly decreasing, on J_i, respectively, and otherwise there exists $t_i \in (x_i, x_{i+1})$ such that S_1' is increasing on (x_i, t_i) and decreasing on (t_i, x_{i+1}) whenever Q is increasing on (x_i, m_i) and decreasing on (m_i, x_{i+1}), and analogously in the case decreasing-increasing;*

(PC$_2$) *for $i = 1, n-1$, S_1' is increasing (decreasing) on J_i if Q is increasing (decreasing) on J_i.*

Finally, S_1 is shape-preserving if it is at the same time PM and PC.

If S_1 is an interpolating function, then

$$\int_{x_i}^{x_{i+1}} P_1 = \int_{x_i}^{x_{i+1}} S_0', \qquad 1 \leq i \leq n-1, \tag{1}$$

where $P_1 := S_1'$. So we have the following.

LEMMA 1. *If S_1 is an interpolating function, then (1) holds with $P_1 = S_1'$. Conversely, if $P_1 \in C^0(J)$ verifies (1), then $S_1(x) := y_1 + \int_{x_1}^{x} P_1$ is an interpolating function.*

A first application of this lemma may be the following. Assume that we wish to construct a PM interpolating quadratic spline in $C^1(J)$, with knots only at $x_2, \ldots, x_{n-1}$. Let $n = 4$, and suppose $0 < d_1 = d_3 \ll d_2$. Applying a visual interpretation of the lemma, we note that such a construction is not possible. Analogously, it is not possible to get a PC interpolating quadratic spline in $C^1(J)$ in the case $n = 5$, $d_1 < d_2 \ll d_3 < d_4$. This drawback is overcome if we let S_1 have one additional knot in each (x_i, x_{i+1}). This fact is a consequence of Lemma 1 and the following.

LEMMA 2. *Let $u, v, z, a_0, b_0, b_1 \in \mathbb{R}$, $u < z < v$. Then there exists a unique broken line $P \in C^0([u, v])$, with a knot only at z, such that*

$$P(u) = a_0, \qquad P(v) = b_0, \qquad \int_u^v P = b_1.$$

PROOF. Let $F(y) := \int_u^v P_y$, $y \in \mathbb{R}$, where P_y is the broken line in $\mathcal{C}^0([u,v])$, with a knot only at z, that verifies $P_y(u) = a_0$, $P_y(v) = b_0$, $P_y(z) = y$. Now the lemma follows from the fact that F is a strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$. This completes the proof.

With a similar geometrical proof, we can get the following result.

LEMMA 3. *Let $u, v, a_0, b_0, b_1 \in \mathbb{R}$, $u < v$. Suppose that $(b_0 - d)(d - a_0) > 0$, where $d := b_1/(v-u)$. Then there exists a unique $z \in (u,v)$ and a unique broken line $P \in \mathcal{C}^0([u,v])$, with a knot only at z, such that $P(u) = a_0$, $P(v) = b_0$, $P(z) = d$, $\int_u^v P = b_1$.*

For $m \in \mathbb{N}$, let $\mathcal{S}_m = \mathcal{S}_m(\Pi_1, \dots, \Pi_n)$ denote the class of interpolating splines in $\mathcal{C}^m(J)$ for which $S_m^{(m-j)}(x_i) = a_{j,i}$, $1 \le i \le n$, $0 \le j \le m-1$, and such that $S_m^{(m)}$ is a broken line with knots only at $x_2, \dots, x_{n-1}$ and at m fixed but arbitrary values in each (x_i, x_{i+1}). The dependence of $\mathcal{S}_m$ on these intermediate knots and on the arbitrary *boundary values* $a_{j,i}$ is not indicated. Due to Lemmas 1 and 2, $\mathcal{S}_1 \ne \emptyset$. Moreover, using Lemmas 1–3 it is very easy to construct a shape-preserving spline $S_1 \in \mathcal{S}_1$. Indeed, after a suitable selection of the boundary values $S_1'(x_i)$, $1 \le i \le n$, the restriction of S_1' on each J_i is computed matching areas of triangles. It is always possible to determine boundary values in such a way that the resultant interpolating spline be shape-preserving (see [6] for a similar treatment of this case). We do not dwell now upon this aspect because it is included in the less simple case $m = 2$, which we shall consider in detail. First we shall prove that for all $m \in \mathbb{N}$, $\mathcal{S}_m \ne \emptyset$. This fact is a consequence of Theorems 1 and 2 below. Let $u, v \in \mathbb{R}$, $u < v$. If $P \in \mathcal{C}^0([u,v])$, then we define the functions on $[u,v]$, $I_0(P) \equiv P$ and $I_j(P,u)(x) := \int_u^x I_{j-1}(P)$, $1 \le j \le m$.

THEOREM 1. *Let $u, v, z_1, \dots, z_m \in \mathbb{R}$, $u < z_1 \cdots < z_m < v$, and let a_0, b_j be $m+2$ arbitrary constants, $0 \le j \le m$. Then there exists a unique broken line $P \in \mathcal{C}^0([u,v])$, with knots only at $z_1, \dots, z_m$, such that*

$$P(u) = a_0, \qquad I_j(P)(v) = b_j, \quad 0 \le j \le m. \tag{2}$$

PROOF. It is by induction on m. For $m = 1$, it is Lemma 2. Assume $m > 1$ and also that the theorem is valid for the knots $z_2, \dots, z_m$. By the inductive hypothesis, for all $y \in \mathbb{R}$ there exists a unique broken line $P_y \in \mathcal{C}^0([u,v])$, with knots only at $z_1, z_2, \dots, z_m$, satisfying $P_y(u) = a_0$, $P_y(z_1) = y$, $I_j(P_y)(v) = b_j$, $0 \le j \le m-1$. The map $y \mapsto I_m(P_y)(v)$ represents a one to one map from $\mathbb{R}$ onto $\mathbb{R}$. Indeed, let $y_1 < y_2$. The function $I_{m-1}(P_{y_2} - P_{y_1})$ has 0 strong sign changes on $[u,v]$. Otherwise, Rolle's Theorem shows that $P_{y_2} - P_{y_1}$ would have at least m strong sign changes, which is impossible. Then it is obvious that $I_{m-1}(P_{y_2} - P_{y_1}) \ge 0$, $I_{m-1}(P_{y_2} - P_{y_1})(z_1) \to +\infty$ for a fixed y_1 and $y_2 \to +\infty$, as well as for a fixed y_2 and $y_1 \to -\infty$. So $I_m(P_{y_2})(v) - I_m(P_{y_1})(v) = I_m(P_{y_2} - P_{y_1})(v) > 0$ and $|I_m(P_y)(v)| \to \infty$ as $|y| \to \infty$. Therefore, there exists a unique $y_0 \in \mathbb{R}$ such that $P_{y_0}(u) = a_0$ and $I_j(P_{y_0})(v) = b_j$, $0 \le j \le m$. So the proof is complete.

Other proofs of Theorem 1 can be given. For instance, resorting to a basis of the linear space of all broken lines in $\mathcal{C}^0([u,v])$, with fixed knots only at $z_1, \dots, z_m$. After choosing a basis, note that we can write (2) as a $(m+2) \times (m+2)$ *nonsingular* linear system. This can be seen either showing that the null vector is the unique solution of the associated homogeneous system, or computing the matrix of the system.

THEOREM 2. *Let $u < z_1 < \cdots < z_m < v$, and let a_j, c_j be $2m+2$ arbitrary constants, $0 \le j \le m$. Then there exists a unique spline S in $\mathcal{C}^{(m)}([u,v])$ satisfying $S^{(m-j)}(u) = a_j$, $S^{(m-j)}(v) = c_j$, $0 \le j \le m$, and such that $S^{(m)}$ is a broken line with knots only at $z_1, \dots, z_m$.*

PROOF. If Q_0 is the polynomial of degree not greater than $m-1$ for which $Q_0^{(m-j)}(u) = a_j$, $1 \le j \le m$, then $S = Q_0 + I_m(P)$, where P is the unique broken line in $\mathcal{C}^0([u,v])$, with knots only at $z_1, \dots, z_m$, that satisfies $P(u) = a_0$ and $I_j(P)(v) = c_j - Q_0^{(m-j)}(v)$, $0 \le j \le m$. This concludes the proof.

From Theorem 2, it is clear that there exists a unique spline in $\mathcal{S}_m$ for each set of arbitrary $m(n-1)$ intermediate knots and arbitrary mn boundary values.

3. CONSTRUCTION OF S_2

We shall obtain a spline $S_2 \in \mathcal{S}_2$ for which the properties (i)–(iv) hold. The form of determining the boundary values $S_2'(x_i), S_2''(x_i)$, $1 \le i \le n$, will be seen at the end of the section. Assuming that they have been already determined, with this information we construct the restriction of S_2, say $S = S_2[i] \in \mathcal{C}^2(J_i)$, on an arbitrary J_i, $1 \le i \le n-1$. Let

$$\alpha = \alpha_i := \frac{(d_i - S'(x_i))}{\delta_i}, \qquad \beta = \beta_i := \frac{(S'(x_{i+1}) - d_i)}{\delta_i},$$
$$a_0 = a_0[i] := S''(x_i), \qquad b_0 = b_0[i] := S''(x_{i+1}),$$

where $(x_{i+1} - x_i)/2 =: \delta_i = \delta$. According to Theorem 2,

$$S(x) = y_i + S'(x_i)(x - x_i) + I_2(P, x_i)(x),$$

where $P \in \mathcal{C}^0(J_i)$ is a broken line, with knots only at z_1, z_2, that satisfies

$$P(x_i) = a_0, \qquad P(x_{i+1}) = b_0, \tag{3}$$
$$I_1(P)(x_{i+1}) := I_1(P, x_i)(x_{i+1}) = S'(x_{i+1}) - S'(x_i) = (\alpha + \beta)\delta, \tag{4}$$
$$I_2(P)(x_{i+1}) := I_2(P, x_i)(x_{i+1}) = 2\,(d_i - S'(x_i))\,\delta = 2\alpha\delta^2. \tag{5}$$

The next job is to calculate P with a suitable selection of the knots in such a way that S_2 becomes shape-preserving. Its construction can be classified in two main schemes. The following conditions will be always valid:

$$0 \le a_0\alpha \le \alpha^2, \qquad 0 \le b_0\beta \le \beta^2. \tag{6}$$

SCHEME I. *$\alpha\beta > 0$, and at least one of the equalities, $a_0 = \alpha$, $b_0 = \beta$, is also valid.*

There are two possible constructions, I(A) and I(B). In I(A) P has no zeros, and in I(B) $P(x)$ may only be zero either at $x = x_i$, or at $x = x_{i+1}$.

I(A). $a_0 = \alpha$, $b_0 = \beta$, $5\min\{|\alpha|, |\beta|\} - 3\max\{|\alpha|, |\beta|\} \ge 0$. If $\alpha = \beta$, then $P \equiv \alpha$. Assume $\alpha \ne \beta$. Let R_1, R_2 be the linear functions that satisfy

$$R_1(x_i) = \frac{5\alpha - 3\beta}{2}, \qquad R_1(m_i) = \alpha,$$
$$R_2(x_{i+1}) = \frac{5\beta - 3\alpha}{2}, \qquad R_2(m_i) = \beta.$$

Consider the family of broken lines $P_{z_1} \in \mathcal{C}^0(J_i)$, with intermediate knots only at $z_1 = z_1[i]$, $x_i < z_1 < m_i$, and at $z_2 = z_2[i] = x_i + x_{i+1} - z_1$, that satisfy

$$P_{z_1}(x_i) = \alpha, \qquad P_{z_1}(x_{i+1}) = \beta,$$
$$P_{z_1}(z_1) = R_1(z_1), \qquad P_{z_1}(z_2) = R_2(z_2).$$

By construction, $I_1(P_{z_1})(x_{i+1}) = (\alpha + \beta)\delta$. If $\alpha < \beta$, then we have

$$I_1(P_{z_1})(x) < I_1(P_{z_1'})(x), \qquad \text{if } z_1 < z_1' \text{ for } x \in (x_i, x_{i+1}).$$

This may be shown by geometrical arguments. In Figure 1, we provide a sketch. Observe that the area of the triangle of vertices (x_i, α), $(z_1', R_1(z_1'))$, and $(m_i, \alpha + \beta/2)$ is smaller than the

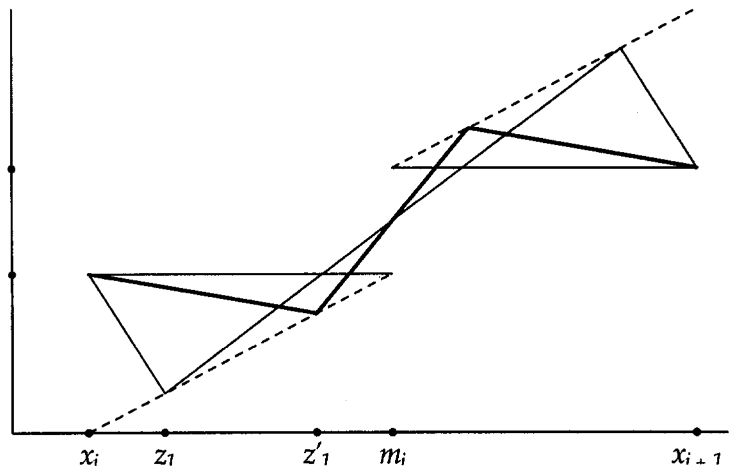

Figure 1. $0 < \alpha < \beta$, where α and β are marked in the vertical axes.

area of the triangle of vertices (x_i, α), $(z_1, R_1(z_1))$, and $(m_i, \alpha + \beta/2)$. This implies the above inequality, taking also into account that $I_1(P_{z_1})(x_{i+1}) = I_1(P_{z'_1})(x_{i+1})$.

The reader is encouraged to sketch a diagram of the remaining schemes that are here regarded. After a simple computation it is shown that

$$\lim_{z_1 \to x_i} I_2(P_{z_1})(x_{i+1}) < 2\alpha\delta^2 < \lim_{z_1 \to m_i} I_2(P_{z_1})(x_{i+1}),$$

whence, there exists a unique value of z_1, say z_1^*, for which $I_2(P_{z_1^*})(x_{i+1}) = 2\alpha\delta^2$. So $P = P_{z_1^*}$ satisfies (3)–(5). If $\beta < \alpha$, then the conclusions are analogous. Solving the equation in (5) for the variable z_1, we obtain $z_1^* = x_i + (7 - \sqrt{33})\delta/4$, $z_2^* = x_{i+1} - (7 - \sqrt{33})\delta/4$ and

$$\begin{aligned} P(x_i) &= \alpha, \qquad P(x_{i+1}) = \beta, \\ P\left(z_1^*\right) &= \frac{\alpha+\beta}{2} - \frac{\left(3\sqrt{33}-5\right)(\beta-\alpha)}{8}, \\ P\left(z_2^*\right) &= \frac{\alpha+\beta}{2} + \frac{\left(3\sqrt{33}-5\right)(\beta-\alpha)}{8}. \end{aligned}$$

I(B). If Scheme I holds but I(A) does not, then P is obtained as follows. We first suppose either $|\alpha| < |\beta|$, or $\alpha = \beta = a_0$. Let R be the linear function that satisfies $R(x_i) = \alpha$, $R(x_{i+1}) = 0$. Consider the family of broken lines $P_{w_1} \in C^0(J_i)$, with knots only at $w_1 = w_1[i] \in (x_i, x_{i+1})$ and at $w_1 + x_{i+1}/2$, such that

$$\begin{gathered} P_{w_1}(x_i) = a_0, \qquad P_{w_1}(x_{i+1}) = b_0, \qquad P_{w_1}(w_1) = R(w_1), \\ I_1(P_{w_1})(x_{i+1}) = (\alpha+\beta)\delta. \end{gathered}$$

It is easy to show that $|I_2(P_{w_1})(x_{i+1})| < |I_2(P_{w'_1})(x_{i+1})|$ if $w_1 > w'_1$, and furthermore,

$$\lim_{w_1 \to x_{i+1}} |I_2(P_{w_1})(x_{i+1})| < 2|\alpha|\delta^2 < \lim_{w_1 \to x_i} |I_2(P_{w_1})(x_{i+1})|.$$

Hence, there exists a unique $P = P_{w_1^*}$ that satisfies (3)–(5). Solving the equation in (5) for the variable w_1, we get

$$w_1^* = x_i + \frac{(6\beta - 4\alpha)\delta}{3\beta}, \qquad \text{if } \alpha = a_0 \quad \text{and} \quad b_0 = 0,$$

and otherwise

$$w_1^* = x_i + \frac{A\delta + \operatorname{sgn}(\alpha)\left(A^2\delta^2 - 4(a_0 - b_0 - \alpha)(3\beta - b_0 - 2\alpha)\delta^2\right)^{1/2}}{\alpha + b_0 - a_0},$$

where $A := 3a_0 + 2b_0 - 3\alpha - 3\beta$. The broken line P is determined by

$$P(x_i) = a_0, \qquad P(x_{i+1}) = b_0,$$
$$P(w_1^*) = \frac{\alpha\,(x_{i+1} - w_1^*)}{2\delta},$$
$$P\left(\frac{w_1^* + x_{i+1}}{2}\right) = \frac{(-b_0 + \alpha + 2\beta)\delta + (b_0/2 - a_0)(w_1^* - x_i) + \alpha(w_1^* - x_i)^2/4\delta}{x_{i+1} - w_1^*}.$$

If $|\beta| < |\alpha|$, or $\beta = \alpha = b_0$, then P is computed transforming this condition to the previous one by "reflection with respect to the y axis" of the scheme for S''. So the knots w_1 and w_2 are now

$$w_1 = \frac{x_i + w_2}{2},$$
$$w_2 = x_{i+1} - \frac{(6\alpha - 4\beta)\delta}{3\alpha}, \qquad \text{if } \beta = b_0 \quad \text{and} \quad a_0 = 0,$$

and

$$w_2 = x_{i+1} - \frac{B\,\delta + \operatorname{sgn}(\beta)\left(B^2\delta^2 - 4(b_0 - a_0 - \beta)(3\alpha - a_0 - 2\beta)\delta^2\right)^{1/2}}{\beta + a_0 - b_0}$$

otherwise, where $B := 3b_0 + 2a_0 - 3\beta - 3\alpha$. The broken line P is determined by

$$P(x_i) = a_0, \qquad P(x_{i+1}) = b_0,$$
$$P(w_1) = \frac{(-a_0 + \beta + 2\alpha)\delta + (a_0/2 - b_0)(x_{i+1} - w_2) + \left(\beta(x_{i+1} - w_2)^2\right)/4\delta}{w_2 - x_i},$$
$$P(w_2) = \frac{\beta(w_2 - x_i)}{2\delta}.$$

We remark that the method used in Scheme I(B) is also applicable to the case of Scheme I(A). The symmetrical procedure applied in I(A) has the advantage of allowing us a more direct estimate of the error bound.

SCHEME II. $\alpha\beta \le 0$. Put $z_1 = x_i + \delta/2$, $z_2 = x_i + 3\delta/2$. The broken line P that satisfies (3)–(5) is uniquely determined by (3) and the conditions

$$P(z_1) = \frac{32\alpha - 7a_0 + 3b_0 - 16\beta}{12},$$
$$P(z_2) = \frac{3a_0 - 16\alpha + 32\beta - 7b_0}{12}.$$

Due to (6) it follows that $P(z_1) \le 0 \le P(z_2)$ whenever $\alpha \le 0 \le \beta$, and $P(z_2) \le 0 \le P(z_1)$ whenever $\beta \le 0 \le \alpha$, whence in either case P may have at most one strong sign change on J_i.

We point out that the computation of P required, as a maximal complexity, to solve roots of quadratic equations in Scheme I.

Determination of the Boundary Values

Normally, $S_2'(x_i)$ is given by $Q(x_i)$, $1 \le i \le n$. This is not a strict rule, but so is the condition $\min\{d_{i-1}, d_i\} \le S_2'(x_i) \le \max\{d_{i-1}, d_i\}$ for $2 \le i \le n-1$.

(a$_1$) The spline $S_2^* \in \mathcal{S}_2$, whose restriction on J_i is obtained according to Schemes I and II, with

$$S'(x_i) = Q(x_i), \qquad S'(x_{i+1}) = Q(x_{i+1}), \qquad S''(x_i) = \alpha_i, \qquad S''(x_{i+1}) = \beta_i,$$

is a PC interpolating spline satisfying Condition (iii) in Section 1.

(a$_2$) The following optional modification may be applied provided it is not in contradiction with other requirements. In the case, $d_{i-1} \neq d_i = d_{i+1}$, put $S'(x_i) = d_i$, $S''(x_i) = \alpha_i = 0$. In the symmetrical case, $d_{i-1} = d_i \neq d_{i+1}$, put $S'(x_{i+1}) = d_i$, $S''(x_{i+1}) = \beta_i = 0$. Both modifications cannot be simultaneously applied on J_i and on J_{i-1} in the case $d_{i-2} = d_{i-1} \neq d_i = d_{i+1}$.

If in addition, a PM interpolating spline is desired, i.e., satisfying (ii) and (iii) in Section 1, then some corrections have to be done in the following cases.

(b$_1$) If $Q(x_1) < 0 < d_1$, then put $S'(x_1) = 0$, $S''(x_1) = \alpha_1$. Proceed analogously when $Q(x_1) > 0 > d_1$. In the symmetrical cases, $d_{n-1} > 0 > Q(x_n)$, or $d_{n-1} < 0 < Q(x_n)$, put $S'(x_n) = 0$, $S''(x_n) = \beta_{n-1}$.

(b$_2$) If $0 = d_i < d_j$, or $0 = d_i > d_j$, $i = 1, j = 2$, or $i = n-1, j = n-2$, then put $S'(x_i) = S'(x_{i+1}) = S''(x_i) = S''(x_{i+1}) = 0$.

(b$_3$) If $d_{i-1} \geq d_i \geq 0$, $d_{i+1} \geq d_i$, $2 \leq i \leq n-2$, then put

$$S'(x_j) = \min\{Q(x_j), 2d_i\}, \qquad j = i, i+1, \qquad S''(x_i) = \alpha_i, \qquad S''(x_{i+1}) = \beta_i.$$

In the symmetrical case, $d_{i-1} \leq d_i \leq 0$, $d_{i+1} \leq d_i$, put

$$S'(x_j) = \max\{Q(x_j), 2d_i\}, \qquad j = i, i+1, \qquad S''(x_i) = \alpha_i, \qquad S''(x_{i+1}) = \beta_i.$$

Now it is easy to prove that the construction of S on J_i according to Scheme II is such that S' has 0 strong sign changes on J_i.

For $2 \leq i \leq n-1$, observe that if a correction is needed at x_i as the left extremal point of J_i, then there is no need of other correction at x_i as the right extremal point of J_{i-1}, and analogously in the symmetrical situation. So these modifications, if needed, are always globally compatible.

REMARK. The way of determining the boundary values responds to the fact of obtaining a shape-preserving interpolating spline satisfying (i)–(iv) in Section 1. However, and unless experimental arguments imply the contrary, such a spline might not be appropriate to explain the behaviour of the data. For example, if $d_{i-1}d_{i+1} > d_i = 0$, there may not be a valid reason to compel $S' \equiv 0$ on J_i. Assuming that the data come from a function $f \in \mathcal{C}^3(J)$, the PC interpolating spline S_2^* could yield more reliable information about f than any rigorous shape-preserving interpolating spline. This is a consequence of the fact that Q is a satisfactory schematic representation of f' on the whole interval J. For instance, this implies that S_2^* always satisfies (iii) in Section 1 without any restriction. On the other hand, in Cases (b$_1$) and (b$_2$) the inequality $|f(x) - S_2(x)| \leq CKh^3$ may not hold for $x \in [x_1, x_3]$ and/or for $x \in [x_{n-2}, x_n]$, and some information about f on J_1 and/or on J_{n-1} is generally needed if we require the same estimate on the whole interval J. The following example enlightens us as to this situation.

Let $n = 3$, $x_1 = -1$, $x_2 = 2$, $x_3 = 3$, $f(x) = x^2$ on $[x_1, x_3]$. We have $K = 0$ in Condition (iii), and indeed $S_2^* \equiv f$. But, obviously, S_2 does not coincide with f on $[x_1, x_2]$, contradicting (iii). However, if we assume that the same data, (x_i, x_i^2), come now from a function f with the property that $f' \geq 0$ on $[x_1, x_2]$ (or $f''(\mu) = 0$ for some $\mu \in [x_1, x_2]$), then we shall prove that S_2 does satisfy (iii).

Likewise, when $0 = d_1 < d_2$ we assume $f''(\mu) = 0$ for some $\mu \in [x_1, x_2]$ in order that S_2 satisfies Property (iii). Analogously, in the other three symmetrical cases.

4. ESTIMATE OF THE ERROR

Suppose that $f \in \mathcal{C}^3(J)$ is an interpolating function. Let $\theta_i \in (x_i, x_{i+1})$ be such that $f'(\theta_i) = d_i$, $1 \leq i \leq n-1$, and assume $f'' \neq 0$ on J_i. Without loss of generality, let $f'' > 0$ on J_i. Due to

Lemma 1 and Taylor's Theorem, we have

$$\int_{\theta_i}^{m_i+\delta_i} \left[f''(\theta_i)(x-\theta_i) + \frac{f'''(\gamma_i(x))}{2}(x-\theta_i)^2\right] dx = \int_{m_i-\delta_i}^{\theta_i} \left[-f''(\theta_i)(x-\theta_i) - \frac{f'''(\gamma_i(x))}{2}(x-\theta_i)^2\right] dx,$$

for some $\gamma_i(x) \in (x_i, x_{i+1})$ for all $x \in [x_i, x_{i+1}] \setminus \{\theta_i\}$. So

$$2f''(\theta_i)(m_i-\theta_i)\delta_i = -\int_{x_i}^{x_{i+1}} \frac{f'''(\gamma_i(x))}{2}(x-\theta_i)^2\, dx,$$

whence,

$$|f''(\theta_i)(m_i-\theta_i)| \le 2K\,\delta_i^2 \le \frac{K\,h^2}{2}. \tag{7}$$

Now let i be fixed, and assume that the following condition holds:

$$\begin{aligned} &\text{if } 1 \le i \le n-2, && \text{then } f'' \ne 0 \text{ on } [x_i, x_{i+2}], \\ &\text{if } 2 \le i \le n-1, && \text{then } f'' \ne 0 \text{ on } [x_{i-1}, x_{i+1}]. \end{aligned} \tag{$*$}$$

So, without loss of generality, suppose $f'' > 0$ in $(*)$.

Assume now that S was calculated on J_i with boundary values given by (a_1) in Section 3. Therefore, if $i \le n-2$, then

$$\begin{aligned} \beta_i &= \frac{f'(\theta_{i+1}) - f'(\theta_i)}{\delta_i + \delta_{i+1}} \\ &= \frac{f'(\theta_{i+1}) - f'(m_{i+1}) + f'(m_i) - f'(\theta_i) + f'(m_{i+1}) - f'(m_i)}{\delta_i + \delta_{i+1}} \\ &= f''(\epsilon_i) + \frac{f''(\eta_i)(m_i-\theta_i)}{\delta_i + \delta_{i+1}} + \frac{f''(\eta_{i+1})(\theta_{i+1}-m_{i+1})}{\delta_i + \delta_{i+1}} \\ &= f''(\epsilon_i) + \frac{f'''(\eta_i')(\eta_i-\theta_i)(m_i-\theta_i)}{\delta_i + \delta_{i+1}} + \frac{f''(\theta_i)(m_i-\theta_i)}{\delta_i + \delta_{i+1}} \\ &\quad + \frac{f'''(\eta_{i+1}')(\eta_{i+1}-\theta_{i+1})(\theta_{i+1}-m_{i+1})}{\delta_i + \delta_{i+1}} + \frac{f''(\theta_{i+1})(\theta_{i+1}-m_{i+1})}{\delta_i + \delta_{i+1}}, \end{aligned} \tag{8}$$

where $\epsilon_i \in (m_i, m_{i+1})$, and $|m_j-\theta_j|, |\eta_j-\theta_j|, |\eta_j'-\theta_j| < \delta_j$ for $j = i, i+1$.

Analogously, if $2 \le i \le n-1$, then

$$\begin{aligned} \alpha_i &= f''(\epsilon_{i-1}) + \frac{f'''(\eta_i')(\eta_i-\theta_i)(\theta_i-m_i)}{\delta_i + \delta_{i-1}} + \frac{f''(\theta_i)(\theta_i-m_i)}{\delta_i + \delta_{i-1}} \\ &\quad + \frac{f'''(\eta_{i-1}')(\eta_{i-1}-\theta_{i-1})(m_{i-1}-\theta_{i-1})}{\delta_i + \delta_{i-1}} + \frac{f''(\theta_{i-1})(m_{i-1}-\theta_{i-1})}{\delta_i + \delta_{i-1}}, \end{aligned} \tag{9}$$

where $\epsilon_{i-1} \in (m_{i-1}, m_i)$, and $|m_{i-1}-\theta_{i-1}|, |\eta_{i-1}-\theta_{i-1}|, |\eta_{i-1}'-\theta_{i-1}| < \delta_{i-1}$.

Furthermore, if $i = 1$, then $\alpha_1 = \beta_1$, and if $i = n-1$, then $\beta_{n-1} = \alpha_{n-1}$. For $x \in [m_i, x_{i+1}]$, $Q(x) = f'(\theta_i) + \beta_i(x - m_i)$. Therefore,

$$\begin{aligned} f'(x) - Q(x) &= f'(m_i) + f''(\omega_i(x))(x-m_i) - f'(\theta_i) - \beta_i(x-m_i) \\ &= f''(\eta_i)(m_i-\theta_i) + (f''(\omega_i(x)) - \beta_i)(x-m_i) \\ &= f''(\theta_i)(m_i-\theta_i) + f'''(\eta_i')(\eta_i-\theta_i)(m_i-\theta_i) \\ &\quad + (f''(\omega_i(x)) - \beta_i)(x-m_i), \end{aligned}$$

where $\omega_i(x) \in (m_i, x_{i+1})$ for all $x \in (m_i, x_{i+1}]$. Thus, using (*), (7), and (8) (or (9) if $i = n-1$), it follows that

$$|f' - Q| \le 3K\,h^2, \qquad \text{on } [m_i, x_{i+1}].$$

The same estimate is obtained on $[x_i, m_i]$.

(E1). Suppose that Scheme I(A) is applied. If $\alpha_i = \beta_i$, then $S' = Q$ on J_i, whence,

$$|f(x) - S(x)| = \left|\int_{x_i}^{x} (f' - Q)\right| = \left|\int_{x}^{x_{i+1}} (f' - Q)\right|, \qquad \text{for all } x \in J_i.$$

Hence,

$$|f - S| \le \frac{3K\,h^3}{2}, \qquad \text{on } J_i.$$

If $\alpha_i \neq \beta_i$, then $2 \le i \le n-2$. Using (*) and (7)–(9) we get $|\beta_i - \alpha_i| \le 8K\,h$. On the other hand, the construction of P in Scheme I(A) implies that $|P - \beta_i| \le 3|\beta_i - \alpha_i|/2$ on $[m_i, x_{i+1}]$, whence, $|P - \beta_i| \le 12K\,h$. On $[m_i, x_{i+1}]$, $S'(x) = S'(x_{i+1}) - \int_x^{x_{i+1}} P$, and $Q(x) = S'(x_{i+1}) - \int_x^{x_{i+1}} \beta_i$. Thus, $|S' - Q| \le 6K\,h^2$, and $|f' - S'| \le |f' - Q| + |S' - Q| \le 9K\,h^2$ on $[m_i, x_{i+1}]$. By symmetry, the same bound holds on $[x_i, m_i]$, whence

$$|f - S| \le \frac{9K\,h^3}{2}, \qquad \text{on } J_i.$$

(E2). In Scheme I(B), when the boundary values are determined according to (a_1) in Section 3, we have

$$\alpha_i\beta_i > 0, \qquad 5\min\{|\alpha_i|, |\beta_i|\} - 3\max\{|\alpha_i|, |\beta_i|\} < 0. \tag{10}$$

Hence, $2 \le i \le n-2$. Replace $f''(\epsilon_{i-1})$ in (9) by $f''(\epsilon_i) + f'''(\epsilon_i')(\epsilon_{i-1} - \epsilon_i)$, or $f''(\epsilon_i)$ in (8) by $f''(\epsilon_{i-1}) + f'''(\epsilon_i')(\epsilon_i - \epsilon_{i-1})$, where $\epsilon_i' \in (m_{i-1}, m_{i+1})$. Using (7)–(9), it follows from (*) and (10) that

$$\max\{|f''(\epsilon_i)|, |f''(\epsilon_{i-1})|\} < 17K\,h. \tag{11}$$

Thus, $|f''(x)| \le |f''(x) - f''(\epsilon_i)| + |f''(\epsilon_i)| < 18.5K\,h$ for $x \in J_i$. Hence, $|f'(x) - f'(\theta_i)| < 18.5K\,h^2$ for $x \in J_i$. On the other hand, by construction of S', $|S'(x) - f'(\theta_i)| \le \max\{|\alpha_i|, |\beta_i|\}\delta_i$. Therefore, using (*), (7)–(9) and (11), we have $|S' - f'(\theta_i)| < 10K\,h^2$ on J_i. So, $|f' - S'| < 28.5K\,h^2$ on J_i. Thus,

$$|f - S| < 19.25K\,h^3, \qquad \text{on } J_i.$$

(E3). Suppose now that $Q(x_1) < 0 < d_1$, $d_2 \neq d_3$. If we wish to obtain a shape-preserving interpolating spline S_2, then the boundary values have to be determined in J_1 according to (a_1), with the correction (b_1) at x_1. Thus, $\alpha_1 = \frac{d_1}{\delta_1}$, $d_1 < d_2$. In this case, we have to assume $f' \ge 0$ on J_1 (see Remark). Under this assumption, a close inspection of f' on $[x_1, x_3]$ shows that $f'' - \alpha_1$ has at least one strong sign change on that interval. Otherwise, f' could not verify $\int_{x_1}^{x_2} f' = 2d_1\delta_1$, $\int_{x_2}^{x_3} f' = 2d_2\delta_2$. Therefore, there exists $\nu \in (x_1, x_3)$ such that

$$\alpha_1 = f''(\nu). \tag{12}$$

So (*), (7), and (8) for $i = 1$, and (12), hold. Now a similar procedure as above for the case (a_1) proves that $|f - S| < M\,K\,h^3$ on $[x_1, x_2]$, $M < 19.25$. Analogous arguments apply for the other three symmetrical positions.

(E4). In all the remaining cases it is easy to see that S is obtained on J_i under conditions that imply the existence of $\mu \in (x_1, x_n)$ such that

$$|\mu - m_i| < \frac{5h}{2} \qquad \text{and} \qquad f''(\mu) = 0.$$

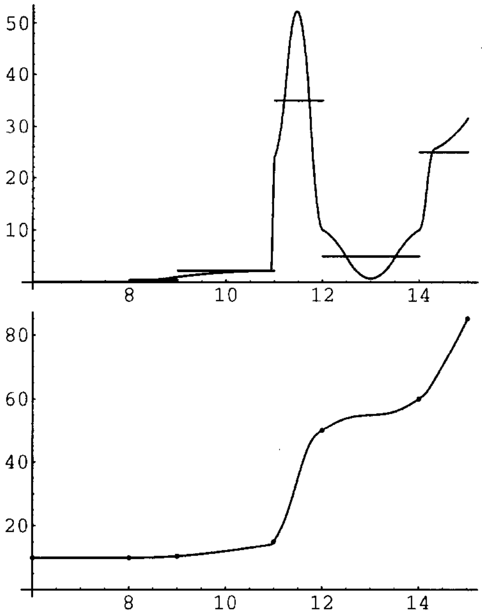

Figure 2.

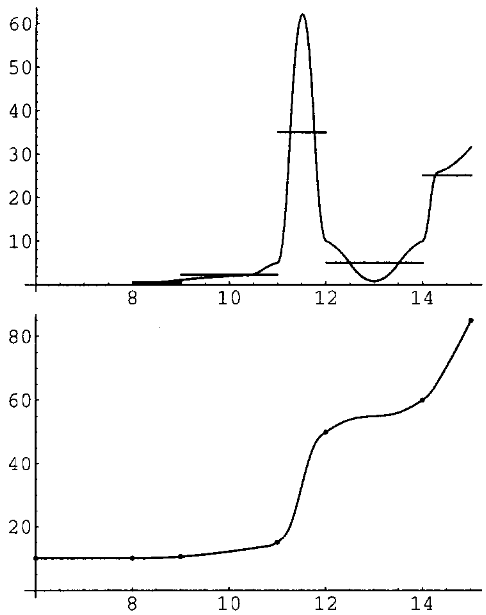

Figure 3.

Therefore, $|f''(x)| \leq 3K\,h$ if $x \in J_i$, $1 \leq i \leq n-1$, and $|f''(x)| \leq 4K\,h$ if $x \in [x_{i-1}, x_{i+2}]$, $2 \leq i \leq n-2$. Hence, for $x \in J_i$, $|f'(x) - f'(\theta_i)| \leq 3K\,h^2$ and

$$\int_{x_i}^{x_{i+1}} |f' - f'(\theta_i)| \leq 3K\,h^3. \tag{13}$$

For $2 \leq i \leq n-2$, the construction of S' according to any scheme implies

$$\min\left\{\max_{x\in J_i}\,[S' - f'(\theta_i)]_+\,, \max_{x\in J_i}\,[S' - f'(\theta_i)]_-\right\} \leq \max\{|f'(\theta_{i+1}) - f'(\theta_i)|\,, |f'(\theta_i) - f'(\theta_{i-1})|\}.$$

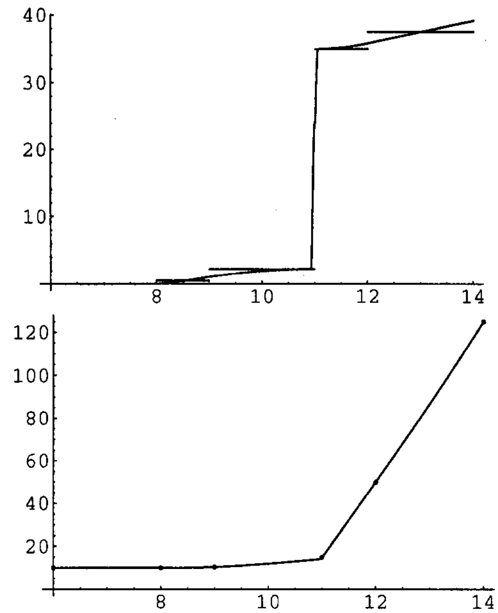

Figure 4.

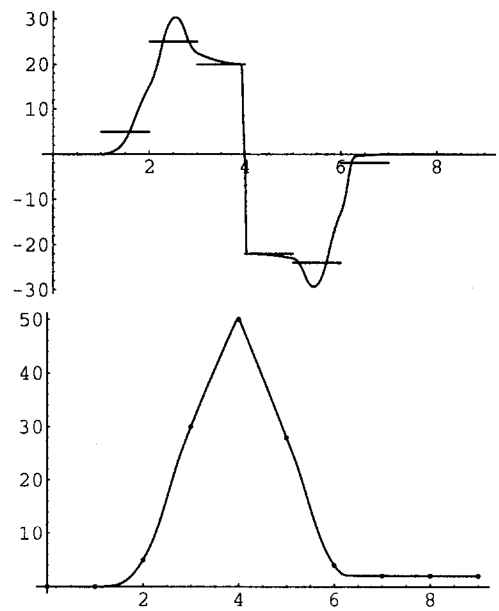

Figure 5.

On the other hand, by Lemma 1, $\int_{x_i}^{x_{i+1}} [S' - f'(\theta_i)]_+ = \int_{x_i}^{x_{i+1}} [S' - f'(\theta_i)]_-$. Then

$$\begin{aligned}\int_{x_i}^{x_{i+1}} |S' - f'(\theta_i)| &= \int_{x_i}^{x_{i+1}} [S' - f'(\theta_i)]_+ + \int_{x_i}^{x_{i+1}} [S' - f'(\theta_i)]_- \\ &\le 2 \int_{x_i}^{x_{i+1}} \min \left\{ \max_{x \in J_i} [S' - f'(\theta_i)]_+ , \max_{x \in J_i} [S' - f'(\theta_i)]_- \right\} \\ &\le 2 \max \left\{ 2\, |f'(\theta_{i+1}) - f'(\theta_i)|\, \delta_i, 2\, |f'(\theta_i) - f'(\theta_{i-1})|\, \delta_i \right\} \le 16K\, h^3.\end{aligned}$$

If $i = 1$, $i = n - 1$, a similar argument leads to the same bound. Finally, using (13), we get for $x \in J_i$ that

$$|f(x) - S(x)| = \left| \int_{x_i}^{x} f' - \int_{x_i}^{x} S' \right| \le \int_{x_i}^{x_{i+1}} |f' - S'| \le 19K\, h^3.$$

EXAMPLES. The following data are taken from [1]:

$$(6,10),(8,10),(9,10.5),(11,15),(12,50),(14,60),(15,85).$$

They have been considered in several papers. In Figure 2, we show the shape-preserving cubic spline S_2 and its derivative above (with the step function S_0'). With regard to the PC spline S_2^* and according to (b_2) and (b_3), respectively, we have changed the boundary values in the intervals $[6,8]$ and $[12,14]$. Thus, $S_2'(6) = S_2''(6) = S_2'(8) = S_2''(8) = 0$ and $S_2'(12) = S_2'(14) = 10$, $S_2''(12) = -5$, $S_2''(14) = 5$.

The two additional knots in the interval $[9,11]$ are very close to the right endpoint. This causes a sharp variation in the derivative. In this case, we can smooth the curve taking a smaller value of $S_2'(11)$. In Figure 3, we show the result. Observe that Condition (iii) of Section 1 remains valid for this new shape-preserving spline, since Case (E4) applies for this spline in the intervals $[9,11]$ and $[11,12]$, where S_2 was modified.

Under the assumption that we wish to preserve convexity, this improvement in the smoothness of the curve would not be possible if the data are in a strict convex position. For instance, consider the following data:

$$(6,10),(8,10),(9,10.5),(11,15),(12,50),(14,125).$$

In Figure 4, we graph S_2. Any attempt to better smoothness in a neighborhood of $x_4 = 11$ will cause to lose convexity in the interval $[9,12]$.

Finally, in Figure 5 we show a graph of the shape-preserving spline S_2 for data which are neither monotone nor convex

$$(0,0),(1,0),(2,5),(3,30),(4,50),(5,28),(6,4),(7,2),(8,2),(9,2).$$

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